ON THE EXTENSION OF THE NECK OF POLYMER SPECIMENS UNDER TENSION

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The phenomenon described below is exhibited by polymer specimens, both amorphous and crystalline structure, subjected to extension under prescribed conditions. When the experiment, as usual, is conducted at a steady rate of deformation (or more exactly, at constant velocity of the clamps securing the specimen in the testing machine), then at first the extension is uniform over the entire specimen and the stress increases proportionally with the deformation (as indicated by the linear segment 1 in the diagram of Fig.1). At a certain instant the homogeneous extension is suddenly interrupted, the specimen exhibits a sharp narrowing, the so called necking,which, however in contrast to the neck occurring in the tensile specimens made of common metals, does not continue to grow thinner and eventually rupture, but having reached a certain thickness propagates along the specimen untill it covers the entire length of the specimen. The edges of the neck propagate along the specimen at a constant rate and exhibit the properties of a solid body in the sense that they preserve their shape (Fig.2). During the propagation of the neck the tensile stress remains constant (segment 2 in Fig.1). After the neck has extended to cover the specimen completely; the deformation proceeds again for some time uniformly along the whole specimen and the tensile stress increases with the deformation approximately according to the linear law (segment 3 in Fig.1). Then at some point of the specimen a sharp narrowing occurs again, this is a "neck of second order"; the phenomenon repeats itself. Under favorable conditions development of necks of a few orders is possible. Fig.3 shows photographs of consecutive extension states of a caprone (*) specimen tested at a temperature of 100°C. Shown from left to right

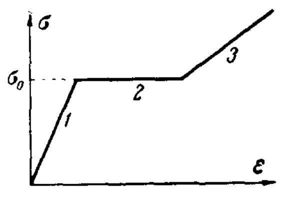

Fig. 1

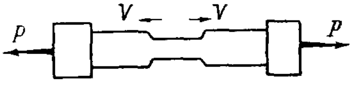

Fig. 2

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*) The experiment was performed by V.I. Shobolova of the Mechanics Institute, Division of plasticity, Moscow State University.

are (a) the original specimen, (b) a specimen in which the neck of the first order extends over part of its length, (c) a specimen in which the neck of the first order has covered the entire length and the neck of the second order can be seen at the top, (d) specimen completely covered by the neck of the second order.

The phenomenon of the characteristic occurence and development of neck in polymer specimens in extension, which received the name of cold drawing was first discovered by Carothers and Hill [1]. They were followed by a number of investigators. Miklowitz [2 and 3], who performed experiments on nylon specimens should be particularly cited. The principal contributions to the investigation of the phenomenon were introduced by Kargin and Sogolova [4 and 5], who showed, precisely and in detail, the connection between formation and development of the neck, and changes in the macromolecular structure of crystalline polymers (*). The experimental and the qualitative theoretical investigations of the formation and development of the neck in amorphous polymers were given in the works of Iu.S. Lazurkin (**). The wealth of experimental results and qualitative arguments included in these works are used in the present paper which also includes an attempt to construct a theory of the formation and development of the neck of polymers subjected to cold drawing.

Fig.3

The theoretical consideration given below is based on the statement widely confirmed by experiments, that during the process, orientational deformations of the macromolecular structure elements of the material take place. It is assumed that the speed of this process of orientational deformation depends strongly on the stresses acting at the given point in the material, so that the speed of the process in the wider region of the specimen is negligibly small compared with that in the transition region of the neck. This fact was conclusively demonstrated by the works of A.P. Aleksandrov (see dissertation by Iu.S. Lazurkin cited above and also the paper [6]). Finally, it is assumed that following the rapture of certain bonds between the macromolecular structural elements, brought about by the action of the applied stress, a corresponding mobility of the elements occurs, resulting

*) Detailed statement of results relating to the change of macromolecular structure of polymers in the neck can be found in the dissertation by Sogolova "Investigation of large deformations of amorphous and crystalline polymers", Moscow, 1963.

**) Lazurkin, Iu.S., The mechanical properties of polymers in vitreous state. Dissertation, M. Inst.fizproblem, Akad.Nauk SSSR, 1954.

in a characteristic transfer of matter which takes place only under sufficiently large stresses.

The proposed theory of the neck propagation appears to be similar in principle to the theory of propagation of the gene given in the paper of Kolmogorov, Petrovskii and Piskunov [7], and in particular to the theory of normal propagation of flames developed in the papers of Zeldovich and Frank-Kamenetskii [8 and 9]. Apparently, this correspondence is not accidental and appears as a result of nontrivial analogy between the considered phenomena; in all cases uniform propagation depends on interaction of the processes of change and transfer of matter.

1. We shall proceed from one-dimensional scheme of the process. The specimen is considered as a rod with variable cross-section area. Distribution of all quantities over the cross-section is assumed uniform (Fig. 4). The x-axis is chosen to coincide with the axis of the rod. The rod is under tension in the axial direction due to the force $P = \sigma_0 S_0$, where S_0 is the cross-section area of the rod at the instant when the uniform deformation ceases and preceding the formation of a neck. The process of development of the neck is slow, it can be therefore considered as quasi-static and the following equation is valid

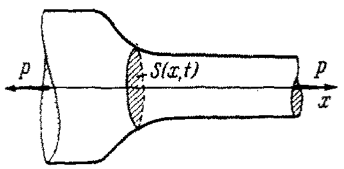

Fig. 4

$$\sigma S = \sigma_0 S_0 = P \tag{1.1}$$

where $\sigma = \sigma(x,t)$ is the stress acting over a cross-section at x and at time t, and $S(x,t)$ is the corresponding cross-section area.

With all differences of microscopic behavior of the process for various polymers, the process always results in transfer of macromolecular elements of the structure of the polymer (*) into a state characterized by higher degree of orientation and corresponding hardening in the direction of drawing. Due to the nonuniformity and known lack of order in the change in the microstructure of the polymer, the process does not take place simultanously in all elements. Therefore, in spite of the fact that the density and the chemical composition of the material may be the same in both states, one can consider the polymer as a bicomponent material consisting of small elements of the matter that has or has not undergone the change in the process of orientational deformation.

We shall utilize a known concept of division of the matter into changed and unchanged particles. In fact, all elements undergo some kind or other of orientational deformation. Nevertheless, due to the nonhomogeneity of the field of microstresses the various elements undergo changes of widely differing degree. We can introduce some critical value of deformation, so that elements exceeding this value are considered as changed.

The microphotograph (Fig.5) due to Sogolova shows a specimen of crystalline polystyrene in the process of neck development. The direction of draw-

*) That is a structure with a characteristic rate greatly exceeding that for molecular material. Such a structure results for crystalline as well as amorphous polymers (see [5, 6 and 10] and also the dissertation by T.I. Sogolova cited above).

ing is shown by the arrow. It can be seen that the material does not deform uniformly; the figure shows the microvolumes of the matter in a state close to the original (darker region) neighboring with microvolumes of transformed matter (lighter regions) having clearly defined fibrous structure extending in the direction of the applied force.

Reasoning this way the material can be considered to consist of two components which we shall call the original and oriented components. The first component changes into the second under the action of the applied stress. It is natural to introduce a concentration factor $n = n(x,t)$ defined as the ratio of volume of the oriented component contained in some sufficiently small region of the total volume of this region, and evaluated at a specific point and time. In compliance with the generally accepted approach to the mechanics of continua this small region (in physical sense an infinitessimal volume) must contain sufficiently large number of the macromolecular elements of the structure so that a meaningful average can be obtained. We shall note that the concentration of the oriented matter n coincides with the supplementary characteristic parameter for solid bodies introduced by Sedov [11] in his general investigation of models of continuous media.

Fig. 5

The cross-section area $S(x,t)$ is uniquely connected with the parameter $n(x,t)$. If the progress is such that as a result of complete change, the value of the concentration parameter of the oriented matter reaches the value n_0 and the area of the cross-section changes α-times, then the change of the area can be assumed to be proportional to the concentration of the oriented matter, so that

$$\frac{S(x,\ t)}{S_0} = 1 - (1-\alpha)\frac{n(x,\ t)}{n_0} \tag{1.2}$$

Here S_0 is the area of the cross-section measured at the instant the homogeneous deformation ceases and before the appearance and development of the neck process.

We shall note here that, in general, the material can exhibit a number of definite changes characterized by specific degrees of orientation, hardening of the changed material and the specific degree of change. The quantity α determines the change taking place and, in general, is different for different changes.

2. Two quations (1.1) and (1.2) correlate three parameters S, σ and n. In order to obtain a complete system of equations it is necessary to take into consideration the kinetics of the change of the matter during the process of development of the orientational deformation.

Let us consider a volume of the material under the conditions of homogeneous tension and deformation. It is natural to define q, the velocity with which the process of change from the original into oriented matter takes place, as a volumetric rate of change per unit volume and unit time. It is evident that the quantity q depends on the degree of the development of the orientational deformation, that is on the concentration of the oriented matter n, as well as on the stress σ

$$q = q\ (n, \sigma) \tag{2.1}$$

The results obtained by Aleksandrov's collaborators (for detailed bibliography see [6]), indicate that the function $q(n,\sigma)$ is very strongly dependent on the stress σ. On the basis of these results it can be assumed that the expression for q has a form of the Arrhenius dependency

$$q = A\left(1 - \frac{n}{n_0}\right)^p \exp\left[-\frac{U - \gamma\sigma}{kT}\right] \tag{2.2}$$

where U is the energy of activation which according to known approximation can be assumed constant, k is the Boltzmann constant, T is the absolute temperature, p and γ are some constants of the material and the coefficient A can be a weak function of σ.

The exact form of the function $q(n,\sigma)$ is not required for the following general investigations. It is sufficient to stipulate that the magnitude of the velocity $q(n,\sigma)$ increases rapidly with the stress σ, that is that in the process of cold drawing the velocity of transportation of the matter in the wide part of the specimen is infinitly small compared to the velocity in the region of the neck.

We shall note that the hypothesis of the strong dependence of the velocity of development of orientational deformation and structural transformation on the stress was one of the two main assumptions forming the basis of the qualitative description of the neck propagation process proposed by Lazurkin (*). It is not intended to discuss here this process as a whole. We shall note only that the second hypothesis of Lazurkin and its further development basically differ from that given below. Some necessary definition of this hypothesis in a more precise form shall be introduced in what follows.

We shall consider now the formulation of the second hypothesis. In the process of the development of orientational deformation in any polymer,

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*) See footnote on p.1265.

crystalline or amorphous, at any rate, rapture of bonds between elements constituting the macromolecular structure of the polymer takes place.

The fact that bonds are ruptured is born out by experimental evidence where appearance of numerous microscopic cracks and discontinuities are observed. This was definitely proved by Lazurkin (*), who performed a series of experiments on cold drawing of amorphous polymers in media of various compositions and discovered a significant influence of the composition of the medium on the development of the neck. (The composition of the medium was such that it could influence the development of the neck only by cohesive forces acting on the surfaces on the microscopic cracks and discontinuities, i.e. by changing the density of the surface energy).

The relative displacement of the elements of the macromolecular structure is increased by the rapture of bonds. The nonuniformity of the imposed stress field, on the microscopic scale, gives rise to the relative displacements of the elements of the micromolecular structure. By this token, particles in a given section do not move as a unit together with the plane of the section which moves at a mean volumetric velocity, instead, a certain transfer of matter through the plane takes place. Two transfer processes which account for the flow of the oriented matter through the plane of the section can be visualized. Firstly, the transfer of the oriented matter takes place in the direction opposite to the gradient of the stress. It is natural to call this process "strain-diffusion". Secondly, the transfer of the oriented matter takes place in direction opposite to the gradient of its concentration. It should be pointed out that although the process is formally similar to diffusion of oriented matter it is not caused by thermal effects whose contribution is negligibly small, but by microstresses. Thus, the thermodinamic forces controlling the transfer of oriented matter are the gradients of stress and of concentration (**).

The respective flow of the oriented matter Q_1 and Q_2 are given by the following

$$Q_1 = -B\frac{\partial \sigma}{\partial x}, \qquad Q_2 = -D\frac{\partial n}{\partial x} \tag{2.3}$$

it is understood that the coefficients B and D, in general, strongly depend on the stress. At present, the information pertaining to the diffusion and strain-diffusion flows is insufficient to determine their relative magnitudes and therefore in the following both flows will be considered.

The second hypothesis can be restated as follows: under the action of the applied stress field a strain-diffusion and diffusive transfer of oriented matter takes place. The total flow of the oriented matter Q per unit time and unit area of the cross-section is given by

$$Q = -B(\sigma)\frac{\partial \sigma}{\partial x} - D(\sigma)\frac{\partial n}{\partial x} \tag{2.4}$$

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*) See footnote on p. 1265.

**) In this work only one-dimensional model is considered. In the multidimensional case, of course, the maximum tangential stress enters all the relationships.

The above descriptions of the appearance of mobility of the elements of oriented matter and their transfer are qualitatively supported by recent experimental investigations of Kargin. Preliminary estimates of the magnitudes of the diffusion and strain-diffusion coefficients indicate that they are small. Nevertheless, the combined effect of these small quantities is sufficient to account for the observed effect.

Composing in the usual manner the equation of continuity for the oriented matter, we obtain

$$\frac{\partial Sn}{\partial t}+\frac{\partial Snu}{\partial x}=q(n,\sigma)S+\frac{\partial}{\partial x}S\left[B(\sigma)\frac{\partial\sigma}{\partial x}+D(\sigma)\frac{\partial n}{\partial x}\right] \tag{2.5}$$

where $u(x,t)$ is the velocity of the section x at the instant t. The overall equation of continuity based on the assumption that the density remains constant has the form

$$\frac{\partial S}{\partial t}+\frac{\partial Su}{\partial x}=0 \tag{2.6}$$

Combining Equations (2.5) and (2.6), we obtain

$$S\frac{\partial n}{\partial t}+Su\frac{\partial n}{\partial x}=q(n,\sigma)S+\frac{\partial}{\partial x}S\left[B(\sigma)\frac{\partial\sigma}{\partial x}+D(\sigma)\frac{\partial n}{\partial x}\right] \tag{2.7}$$

In this manner the process of orientational deformation of a specimen subjected to cold drawing is described by the system of two differential equations (2.6) and (2.7) and two end conditions (1.1) and (1.2). Together with the corresponding initial and boundary conditions the system completely determines the process.

3. We shall now consider a uniform propagation of the neck. Once more, we note, that there exists a deep analogy between this process and the propagation of the gene [7] and the normal propagation of flames [8 and 9] due to the generality characterizing the approach to these problems. The uniform propagation of the neck is described by the solution of the system of equations obtained above. This solution is invariant under the transformation of time and space coordinates. In order to solve the system we shall consider the specimen to be infinitely long, so that the clamps are positioned at $x = \pm\infty$. We shall assume that the edge of the neck propagates from right to the left so that the material near the right clamp $(x \to +\infty)$ is completely deformed, and on the other side $(x \to -\infty)$ the orientational deformation has not started. In this way, no change takes place in the vicinity of the clamps and there the flow of oriented matter is absent. The clamps are moving apart at a constant rate which causes the edge of the neck to move at a steady velocity V. We shall assume also that in the absolute reference frame the left clamp is fixed and the right moves. We shall use the coordinate $\xi = x + Vt$, moving with the edge of the neck, so that the process is steady. Equations (2.6) and (2.7) become then ordinary differential equations and assume the form

$$\frac{dSu}{d\xi}=0, \qquad Su=\text{const} \tag{3.1}$$

$$Su\frac{dn}{d\xi} = q(n, \sigma) S + \frac{d}{d\xi} S\left[B(\sigma)\frac{d\sigma}{d\xi} + D(\sigma)\frac{dn}{d\xi}\right] \tag{3.2}$$

(The quantity u here and in the following development is preserved for the velocity in the moving coordinate system).

The value of the constant in (3.1) is determined from the conditions at the left clamp

$$S = S_0, \qquad u = V \quad \text{at} \quad \xi \to -\infty$$

so that (3.1) assumes the form

$$Su = S_0 V \tag{3.3}$$

The velocity v, at which the clamps part, is determined as follows

$$v = u(\infty, t) - u(-\infty, t) = \frac{1}{S(\infty, t)} S_0 V - V = \beta V \qquad \left(\beta = \frac{1-\alpha}{\alpha}\right) \tag{3.4}$$

Eliminating n, S and u from Equation (3.2) with the help of Equations (1.1), (1.2) and (3.3) we obtain the equation for the distribution of stress $\sigma(\xi)$

$$n_0 V\frac{d\sigma}{d\xi} = q(n, \sigma)\sigma(1-\alpha) + \sigma^2(1-\alpha)\frac{d}{d\xi} P(\sigma, \sigma_0)\frac{d\sigma}{d\xi} \tag{3.5}$$

where

$$P(\sigma, \sigma_0) = \frac{1}{\sigma}\left[B(\sigma) + D(\sigma)\frac{\sigma_0 n_0}{\sigma^2(1-\alpha)}\right] \tag{3.6}$$

At the left clamp the orientational deformation has not started yet, at the right clamp it has reached completion, therefore at both clamps the flow of oriented matter is equal to zero, and the distribution of stresses $\sigma(\xi)$ must satisfy two boundary conditions

$$\sigma = \sigma_0, \quad \frac{d\sigma}{d\xi} = 0 \quad \text{at} \quad \xi \to -\infty$$
$$\sigma = \frac{\sigma_0}{\alpha}, \quad \frac{d\sigma}{d\xi} = 0 \quad \text{at} \quad \xi \to +\infty \tag{3.7}$$

The independent variable ξ can be eliminated from Equation (3.5) by taking $z = P(\sigma, \sigma_0) d\sigma/d\xi$ and considering σ to be the independent variable. Equation (3.5) is then reduced to the first order equation

$$\frac{dz}{d\sigma} = \frac{n_0 V}{\sigma^2(1-\alpha)} - \frac{qP}{\sigma z} \tag{3.8}$$

conditions (3.7) assume the form

$$z = 0 \quad \text{at} \quad \sigma = \sigma_0, \qquad z = 0 \quad \text{at} \quad \sigma = \sigma_0/\alpha \tag{3.9}$$

It is convenient to change to the following variables

$$\tau = \frac{(\sigma - \sigma_0)}{\sigma_0 \beta}, \qquad u = \frac{\sigma_0 z}{\beta} \qquad \left(\beta = \frac{1-\alpha}{\alpha}\right) \tag{3.10}$$

Then Equation (3.8) and the end condition (3.9) take the form

$$\frac{du}{d\tau} = \frac{n_0 V}{(1-\alpha)(1+\beta\tau)^2} - \frac{\theta(\tau, \sigma_0)}{u}, \qquad \theta = \frac{qP\sigma_0^2}{\beta(1+\beta\tau)} \tag{3.11}$$

$$u = 0 \quad \text{at} \quad \tau = 0, \qquad u = 0 \quad \text{at} \quad \tau = 1 \tag{3.12}$$

It is easy to see that a necessary condition for existence of solutions of the boundary value problem (3.11), (3.12) is

$$\theta\ (0, \sigma_0) = 0, \qquad \theta\ (1, \sigma_0) = 0 \tag{3.13}$$

The physical meaning of these conditions is clearly the absence of change at infinity. Indeed, if for instance the velocity of change was finite at infinity to the left (the wider part of the specimen) then in the short time the process could not longer exist as the neck would have reached the left clamp and all the matter would have been converted into the oriented state.

One would think that the first condition of (3.13) contradicts the exponential relationship of Equation (2.2), i.e. if this relationship is assumed, the first condition of (3.13) will not be satisfied. However it is the exponential dependence of the velocity of transfer that makes it negligibly small in the wide region of the rod compared to that in the region where propagation takes place. Therefore the change of whatever small but finite quantity of matter in the wide region of the rod would necessarily take a very long time, during which the neck propagates over large distances. For specimens of realistic dimensions the preliminary orientational deformation in the wider part of the specimen is infinitely small and conditions (3.13) can always be considered satisfied. (As is well known, a similar situation arises in the theory of combustion [9]).

We shall note now that it is not possible to solve the first order equation (3.11) for an arbitrary set of values of parameters V and σ_0 and satisfy the two end conditions (3.12) at the same time. As is usually the case, the velocity of the separation of clamps v , or for that matter, the neck propagation velocity $V = (1/\beta)v$ is arbitrary. However the value of the parameter $\sigma_0 = \sigma_0^*$, the "critical stress" (*), has a specific significance and has to be chosen so that the sought solution satisfies both end conditions (3.12). The physical meaning of the parameter σ_0 derives from the fact that for a given velocity of clamps the neck forms only when the stress reaches a well defined value. It is evident that for some values of the velocity of drawing there exists no values of $\sigma_0 = \sigma_0^*$ which will give solutions satisfying both end conditions. This means that at such velocities of drawing rupture will occur with no cold drawing taking place.

4. We shall now derive certain conditions under which the critical stress σ_0^* exists and is unique. We assume that $\theta(\tau,\sigma_0) = 0$ not only at $\tau = 0$ but also on the interval $0 \leqslant \tau \leqslant \tau_0 < 1$, and for $\tau_0 < \tau < 1$, the function $\theta(\tau,\sigma_0)$ is a continuously differentiable function of both arguments and monotonously increasing with σ_0. The first assumption, as in the theory of combustion [9 and 12], assures in a certain sense the stability of the process. Under these assumptions the critical stress σ_0^* exists and is unique for all values of V . Accoeding to Equation (3.11) for $\sigma_0 = 0$ we have $\theta\ (\tau, \sigma_0) \equiv 0$, then the solution of Equation (3.11) satisfying the first of conditions (3.12) has the form

$$u_0(\tau) = \frac{n_0 V\tau}{(1-\alpha)(1+\beta\tau)} \tag{4.1}$$

*) "Recrystallization stress" according to accepted terminology [4 and 5].

so that

$$u_0(1) = n_0 V / \beta > 0$$

The family of solutions of Equation (3.11) satisfying the first of the conditions (3.12) for $\sigma_0 \geqslant 0$, is shown in Fig.6. All curves of the family fall below the curve corresponding to $u = u_0(\tau)$ and form two classes. The curves of the first class intersect the ordinate $\tau = 1$ at points corresponding to positive values of u and the curves of the second class intersect the abscissa at points $\tau < 1$ not reaching the ordinate $\tau = 1$. The solution curve dividing the two classes passes through the saddle point $u = 0$, $\tau = 1$ and is the unique sought solution. The corresponding value of the parameter $\sigma_0 = \sigma_0^*$ is the critical stress.

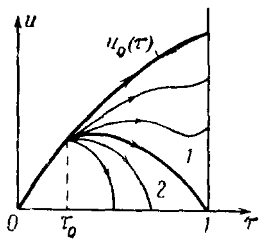

Fig. 6

We shall show the individual steps of the proof of existence and uniqueness of the sought solution of the boundary value problem and the stress σ_0^*.

In order to prove the existence we shall note that if $\sigma_0'' > \sigma_0'$ and $u(\tau, \sigma_0'') > 0$, $u(\tau, \sigma_0') > 0$ then

$$u(\tau, \sigma_0') > u(\tau, \sigma_0'') \quad \text{for} \quad \tau_0 < \tau < 1 \tag{4.2}$$

Differentiating Equation (3.11) with respect to σ_0 gives

$$\frac{dv}{d\tau} = \frac{1}{u^2}\theta(\tau, \sigma_0)\, v - \frac{1}{u}\frac{\partial\theta}{\partial\sigma_0}, \qquad v = \frac{\partial u}{\partial\sigma_0} \tag{4.3}$$

Considering this as a linear equation in v and using the condition $v(\tau_0) = 0$, we obtain

$$v(\tau) = -\exp\left(\int_{\tau_0}^{\tau}\frac{\theta\, d\tau}{u^2}\right)\int_{\tau_0}^{\tau}\exp\left(-\int_{\tau_0}^{\tau}\frac{\theta\, d\tau}{u^2}\right)\frac{1}{u}\frac{\partial\theta}{\partial\sigma_0}\, d\tau < 0 \tag{4.4}$$

From this results the proposed inequality (4.2), since we have

$$u(\tau, \sigma_0'') - u(\tau, \sigma_0') = \int_{\sigma_0'}^{\sigma_0''} v\, d\sigma_0 < 0$$

As direct consequence of the inequality (4.2) and Equation (3.11) integrated between the limits zero and τ, for $u \geqslant 0$ we get

$$u(\tau, \sigma_0) = u_0(\tau) - \int_{\tau_0}^{\tau}\frac{\theta(\tau, \sigma_0)\, d\tau}{u(\tau, \sigma_0)} < u_0(\tau) - \int_{\tau_0}^{\tau}\frac{\theta(\tau, \sigma_0)\, d\tau}{u_0(\tau)} \tag{4.5}$$

so that for any $\tau = \tau_*$ included in the interval $\tau_0 < \tau_* < 1$ the solution of Equation (3.11) and satisfying the first of conditions (3.12) satisfies also the inequality

$$u(\tau_*, \sigma_0) < u_0(\tau_*) - \int_{\tau_0}^{\tau_*}\frac{\theta(\tau, \sigma_0)\, d\tau}{u_0(\tau)} \tag{4.6}$$

From this inequality and the fact that $u_0(\tau)$ is bounded, whereas $\theta(\tau, \sigma_0)$ increases without bounds with σ_0, it follows that for some $\sigma_0 = \sigma_{01}$ the solution $u(\tau, \sigma_0)$ vanishes at the point $\tau = \tau_*$.

We shall now consider $u_1(\tau, \sigma_0)$, the solutions of Equation (3.11) satisfying the second of the conditions (3.12). The point $u = 0$, $\tau = 1$ is a singular point of the system, a saddle point. Through this point pass two solution curves (separatrices). The slope of these curves at the point $\tau = 1$

satisfies the characteristic equation

$$k^2 - \frac{n_0 V\alpha^2}{1-\alpha} k + \theta_\tau'(1, \sigma_0) = 0 \tag{4.7}$$

Since $\theta_\tau'(1, \sigma_0) < 0$, one curve has a positive slope, the other negative. For obvious reasons we choose the curve with the negative slope.

Owing to the continuity of the solution field a curve $u(\tau, \sigma_0)$ for sufficiently small σ_0 approaches the curve $u_0(\tau)$ over the whole interval $\tau_0 \leqslant \tau \leqslant 1$. From the continuity of the solutions $u_1(\tau, \sigma_0)$ it also follows that for a τ_* chosen sufficiently close to $\tau = 1$ and for small σ_0, $u(\tau, \sigma_0) - u_1(\tau_*, \sigma_0) > 0$. However, for $\sigma_0 = \sigma_{01}$ the difference is obviously negative. Again from the continuity of the difference it follows that there exists $\sigma_0 = \sigma_0^*$ for which $u_1(\tau_*, \sigma_0^*) = u(\tau_*, \sigma_0^*)$. This value of the parameter σ_0 has to be determined.

The uniqueness of the solution and of the critical stress can be proved from the very same considerations. Let us assume that two solutions of the boundary value problem exist corresponding to $\sigma_0 = \sigma_0^*$ and $\sigma_0 = \sigma_0^{**} > \sigma_0^*$. Then in the vicinity of $\tau = \tau_0$, according to (4.2), $u(\tau, \sigma_0^{**}) < u(\tau, \sigma_0^*)$. For $\tau_0 < \tau < 1$, the solution $u(\tau, \sigma_0^{**})$ cannot vanish owing to fact that $du/d\tau > 0$ for $u < 0$ such a solution curve cannot reach the point $u = 0$, $\tau = 1$. It follows from this that $u(\tau, \sigma_0^{**})$ vanishes in the interval $0 \leqslant \tau \leqslant 1$ only at $\tau = 0$ and $\tau = 1$. Therefore, form the differentiability of $u(\tau, \sigma_0^{**})$, the inequality (4.2) and the conditions (3.12) results the existence of a value τ_{**} in the interval $\tau_0 < \tau < 1$, such that $|du(\tau, \sigma_0^{**}) / d\tau| < |du(\tau, \sigma_0^*) / d\tau|$. However, this condition is impossible inasmuch as $u(\tau, \sigma_0)$ satisfies Equation (3.11) and $\theta(\tau_{**}, \sigma_0^{**}) > \theta(\tau_{**}, \sigma_0^*)$, or $u(\tau_{**}, \sigma_0^{**}) < u(\tau_{**}, \sigma_0^*)$. The resulting contradiction proves uniqueness.

In the general case, for an arbitrary form of the function $\theta(\tau, \sigma_0)$, the value σ_0^* can be found since the Cauchy problem associated with Equation (3.11) can always be solved for one or the other of the initial values

$$u = \frac{n_0 V \tau_0}{(1-\alpha)(1+\beta\tau_0)} \quad \text{for} \quad \tau = \tau_0, \quad \text{or} \quad u = 0 \quad \text{at} \quad \tau = 1$$

for different values of σ_0. Therefore σ_0^* can be found to any required degree of accuracy by trial and error.

For the practically interesting case when τ_0 has a value close to 1, some simple conditions determining σ_0^* can be obtained from the corresponding development in the theory of combustion [8 and 9]. It is easy to show that

$$u(\tau) < \left(2\int_\tau^1 \theta(\tau, \sigma_0)\, d\tau\right)^{1/2} \qquad u(\tau_0) < \left(2\int_{\tau_0}^1 \theta(\tau, \sigma_0)\, d\tau\right)^{1/2} = F(\sigma_0) \tag{4.8}$$

where $F(\sigma_0)$ is a monotonous, increasing function of σ_0. In this way we find

$$F(\sigma_{01}^*) = \frac{n_0 V \tau_0}{(1-\alpha)(1+\beta\tau_0)} \tag{4.9}$$

where σ_{01}^* is an estimate from below of σ_0^*.

Substituting into the right-hand side of Equation (3.11) the estimate from above of $u(\tau)$, from (4.8) we obtain

$$u(\tau_0) > \left(\int_{\tau_0}^1 2\theta(\tau, \sigma_0)\, d\tau\right)^{1/2} - \frac{\alpha(1-\tau_0) n_0 V}{(1-\alpha)(1+\beta\tau_0)} \tag{4.10}$$

from which it follows that σ_{02}^*, an estimate from above of σ_0^* satisfies Equation

$$F(\sigma_{02}^*) - \frac{\alpha(1-\tau_0) n_0 V}{(1-\alpha)(1+\beta\tau_0)} = \frac{n_0 V \tau_0}{(1-\alpha)(1+\beta\tau_0)} \tag{4.11}$$

For $\tau_0 \to 1$ both estimates coincide resulting in the final relation determining the critical stress $\sigma_0{}^*$

$$F(\sigma_0{}^*) = \frac{n_0 V \alpha}{1-\alpha} \tag{4.12}$$

From (3.11) we have

$$F(\sigma_0) = \left(2\int_{\tau_0}^{1} \theta\ (\tau,\ \sigma_0)\ d\tau\right)^{1/2} = \sigma_0 \left(2\int_{\tau_0}^{1} \frac{qP\alpha d\tau}{(1-\alpha)(1+\beta\tau)}\right)^{1/2} \tag{4.13}$$

For τ_0 approaching unity the above equation assumes the form

$$F(\sigma_0) = \frac{\alpha\sigma_0}{\sqrt{1-\alpha}}\left(2P\left(\frac{\sigma_0}{\alpha}\right)\int_{\tau_0}^{1} q\ (\tau,\ \sigma_0)\ d\tau\right)^{1/2} \tag{4.14}$$

as only $q(\tau,\sigma_0)$ varies rapidly in the vicinity of $\tau \doteq 1$.

If, in particular, Expression (2.2) is assumed for $q(n,\sigma)$, then (considering $(1-\alpha)\gamma\sigma_0 / \alpha kT \gg 1$ in order for $q\ (\tau,\ \sigma_0)$ to be a rapidly varying function in the vicinity of $\tau = 1$) we obtain the following expressions:

$$q \approx A\alpha^p \exp\left[-\frac{U}{kT} + \frac{\gamma\sigma_0}{\alpha kT}\right](1-\tau)^p \exp\left[-\frac{(1-\alpha)\gamma\sigma_0}{\alpha kT}(1-\tau)\right]$$

$$\int_{\tau_0}^{1} q d\tau \approx A\alpha^{2p+1}(1-\alpha)^{-p-1}\exp\left[-\frac{U-\gamma\sigma_0/\alpha}{kT}\right]\left(\frac{kT}{\sigma_0}\right)^{p+1}\Gamma(p+1) \tag{4.15}$$

Correspondingly, the expression determining $\sigma_0{}^*$ becomes

$$V^2\delta = \left[\sigma_0{}^* B\left(\frac{\sigma_0{}^*}{\alpha}\right) + D\left(\frac{\sigma_0{}^*}{\alpha}\right)\frac{n_0\alpha^2}{1-\alpha}\right]\left(\frac{kT}{\gamma\sigma_0{}^*}\right)^{p+1}\exp\left(\frac{\gamma\sigma_0{}^*}{\alpha kT}\right) \tag{4.16}$$

where δ is given by

$$\delta = \frac{n_0{}^2(1-\alpha)^p \exp(U/kT)}{2A\alpha^{2p+1}\Gamma(p+1)} \tag{4.17}$$

Finally we shall note that apart from the problem of uniformly propagating neck as discussed in the above development, there is another significant formulation of the problem of the orientational deformation. Specifically, we shall consider the case in which the specimen is not of constant cross-section but whose cross-section narrows down in some specified manner. Obviously, under loading, the orientational deformation will start in the narrow region. Two possibilities arise. First, if the process of transfer of oriented matter assumes sufficiently high intensity, the narrowing will quickly propagate in both directions and the ensuing process asymptotically approaches the character of neck propagation developed above. The second possibility is that the transfer process will not reach sufficient intensity and the neck will grow thinner until fracture occurs. For a given set of values of the parameters it is the shape of the narrowing or the function describing the variation of the cross-section over the length of the sample that determines which of these two possibilities will take place. In this way a new problem arises, namely to determine what forms of the narrowing will result in uniform propagation and what forms will result in rupture. We shall note that this problem is conceptually similar to the problem of development of chemical reactions studied by Zel'dovich and Gel'fand [12]. Solution of this problem allows to derive the conditions under which the process switches from cold drawing to rupture.

The usefulness of the solution of the problems of uniform propagation of the neck and the development of the orientational deformation is twofold. First, it allows to determine the critical stress and other characteristic parameters of the process. Secondly it provides the foundation for the study of the agreement between the theory of the kinetics of the orientational deformation and the measurements of the critical stress for different velocities of drawing and under different temperatures.

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